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LETTER TO THE EDITOR

Spiral self-avoiding walks on a triangular lattice

K Y Lin

Physics Department, National Tsing Hua University, Hsinchu, Taiwan 300, Republic of China

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Abstract. An exact result is obtained for the number S_n of spiral self-avoiding walks with n steps on a triangular lattice. For $n \to \infty$, S_n increases as $2^{1/4}3^{-7/4}\pi n^{-5/4} \exp[\pi(2n/3)^{1/2}]$.

Spiral self-avoiding walks (SSAW) on a square lattice were first considered by Privman (1983), and solved exactly by Blöte and Hilhorst (1984), Guttmann and Wormald (1984), and Joyce (1984). The SSAW problem can be defined on a triangular lattice such that each bond may either point in the same direction as the preceding one, or in a direction rotated by $+2\pi/3$ (turn left) with respect to it. Similar to the case of the square lattice a SSAW on a triangular lattice in general consists of an outward spiralling part and an inward part.

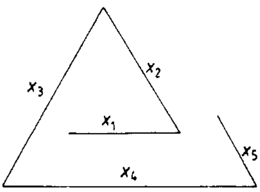

Figure 1. An outward spiralling self-avoiding walk.

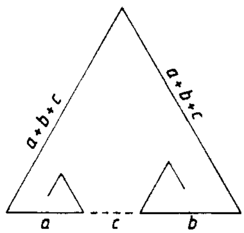

Figure 2. A spiral self-avoiding walk with two longest segments.

Consider first the simpler problem of the subclass of SSAW which only spirals outward (see figure 1). We denote the segment lengths by the integers $x_1, \ldots, x_{m+1}$ where $0 < x_1 < \ldots < x_m$, $x_{m+1} \geqslant 1$. Each walk corresponds one-to-one with a particular partition with distinct parts and therefore we have

$$S_n^* = \sum_{k=0}^{n-1} p(D, k), \tag{1}$$

where S_n^* is the number of n-step SSAWs on a triangular lattice which only spiral outward, and $p(D, k)$ is the number of all partitions with distinct parts. The related generating function is

$$G^*(z) = \sum_{n=1}^{\infty} S_n^* z^n. \tag{2}$$

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It is well known in the theory of partitions (see Andrews 1976, p 5) that $p(D, n) = p(0, n)$ where $p(0, n)$ is the number of all partitions with odd parts and

$$g(z) = \sum_{n\geq 0} p(D, n) z^n = \prod_{m=1}^{\infty} (1+z^m)$$

$$= \sum_{n\geq 0} p(0, n) z^n = \prod_{m=1}^{\infty} (1-z^{2m-1})^{-1}. \tag{3}$$

Consequently we have

$$G^*(z) = z(1-z)^{-1} \prod_{m=1}^{\infty} (1-z^{2m-1})^{-1} \tag{4}$$

and the asymptotic behaviour of S_n^* to leading order is readily determined by Meinardus's theorem (see Andrews 1976, p 89):

$$S_n^* \sim 3^{1/4}(2\pi)^{-1} n^{-1/4} \exp[\pi(n/3)^{1/2}]. \tag{5}$$

We now consider the problem of all SSAW. Each walk has either one or two longest segments. The generating function for all SSAWs with two longest segments (see figure 2) is

$$G_0(z) = \sum_{a,b,c,m,n} z^{3a+3b+2c+m+n} p(D, a, m) p(D, b, n), \tag{6}$$

$$\sum_m p(D, a, m) z^m = (1+z)(1+z^2) \ldots (1+z^{a-1}) \tag{7}$$

where $a \geq 0$, $b \geq 0$, $c \geq 0$ and $p(D, a, m)$ is the number of all partitions with distinct parts such that the largest part is less than a. We have

$$G_0 = \left(\sum_c z^{2c}\right)\left(\sum_{a,m} z^{3a+m} p(D, a, m)\right)\left(\sum_{b,n} z^{3b+n} p(D, b, n)\right)$$

$$= z^2(1-z^2)^{-1} F_3^2 \tag{8}$$

where

$$F_k \equiv 1 + z^k + z^{2k}(1+z) + \ldots + z^{mk}(1+z)(1+z^2) \ldots (1+z^{m-1}) + \ldots$$

$$= \sum_{a,m} z^{ak+m} p(D, a, m). \tag{9}$$

It is simple to prove that

$$F_{k+1}(z) = (z^{-k} - 1)(F_k(z) - 1) \tag{10}$$

where $F_1(z) - g(z)$.

Finally we consider all SSAWs with only one longest segment. They can be classified into four patterns as shown in figure 3. The generating functions are denoted by $G_i(z)$. It is obvious that $G_3 = G_4$ since each walk of pattern three transforms into a walk of pattern four and vice versa by reflection about the vertical line. We have

$$G_1 \quad \sum_{a\geq 0, b\geq 0, c>0, m, n} z^{2a+2b+c+m+n} p(D, a, m) p(D, b, n)$$

$$= z(1-z)^{-1} F_2^2 \tag{11}$$

$$G_2 = \sum_{a>0, b>0, c\geq 0, m, n} z^{2a+2b+3c+m+n} p(D, a, m) p(D, b, n)$$

$$= (1-z^3)^{-1}(F_2 - 1)^2. \tag{12}$$

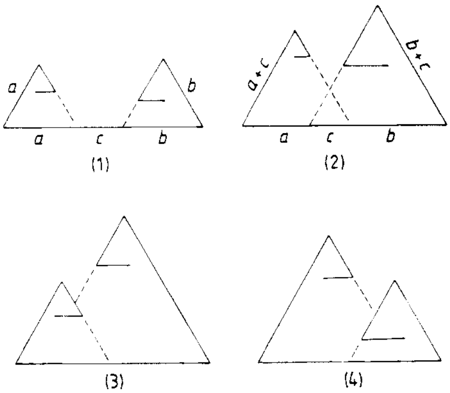

Figure 3. The SSAW with only one longest segment can be classified into four different patterns.

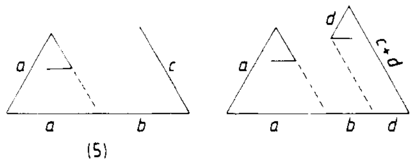

Figure 4. Two subclasses of SSAW.

It is complicated to calculate G_3 directly. It is simpler to calculate first the generating functions of two subclasses of SSAWs shown in figure 4, and then obtain G_3 by substration as follows.

$$G_3 = G_5 + G_6 - G_1 - G_2, \tag{13}$$

where

$$G_5 = \sum_{a\geq 0, b>0, a+b>c\geq 0, m} z^{2a+b+c+m} p(D, a, m)$$
$$= z(1-z)^{-2}F_2 - z^2(1-z)^{-1}(1-z^2)^{-1}F_3, \tag{14}$$

$$G_6 = \sum_{\substack{a\geq 0, b>0, a+b>c>0 \\ d>0, m, n}} z^{2a+b+c+3d} p(D, a, m) p(D, p, n)$$
$$= z^2[(1-z)^{-2}F_2 - (1-z)^{-1}(1-z^2)^{-1}F_3](F_3-1). \tag{15}$$

The generating function for all SSAWs is

$$G(z) = \sum_n S_n z^n$$
$$= G_0 + G_1 + G_2 + G_3 + G_4$$
$$= (2z^5+5z^4+4z^3-2z-1)z^{-4}(1-z)^2(1-z^3)^{-1}g^2(z)$$
$$+2(z^6-3z^4-3z^3+2z+1)z^{-4}(1-z)(1-z^3)^{-1}g(z)$$
$$+(z^7-z^6-z^5+z^4+2z^3-2z-1)z^{-4}(1-z^3)^{-1}$$
$$= z+2z^2+3z^3+5z^4+8z^5+11z^6+17z^7+25z^8+33z^9+\ldots \tag{16}$$

where S_n is the number of all SSAWs with n steps. It follows directly from Meinardus' theorem that the asymptotic behaviour of S_n is determined by

$$8z(1-z)^2(1-z^3)^{-1} \prod_{m=1}^{\infty} (1-z^{2m-1})^{-2} \tag{17}$$

and we have

$$S_n \sim 2^{1/4}3^{-7/4}\pi n^{-5/4} \exp[\pi(2n/3)^{1/2}]. \tag{18}$$

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References

Andrews G E 1976 *The Theory of Partitions* (London: Addison-Wesley)
Blöte H W J and Hilhorst H J 1984 *J. Phys. A: Math. Gen.* **17** L111-5
Guttmann A J and Wormald N C 1984 *J. Phys. A: Math. Gen.* **17** L271-4
Joyce G S 1984 *J. Phys. A: Math. Gen.* **17** L463-7
Privman V 1983 *J. Phys. A: Math. Gen.* **16** L571-3